Radiative Transfer with Anisotropic Scattering and Arbitrary Temperature for Plane Geometry

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Particular solutions to the radiative transport equation are presented for an absorbing, emitting, and anisotropically scattering medium with an arbitrary, but specified, temperature profile; the radiative transport is assumed as one-dimensional and axisymmetric. Homogeneous and particular solutions are obtained from the discrete ordinate form of the radiative transport equation. Derivation of the particular solutions is based upon the method of variation of parameters. The constants associated with the particular solution are expressed explicitly. This work yields a general solution which is applicable to a wide range of problems involving radiative transport in absorbing, emitting, and scattering media. Also, illustrative example problems are presented for media having various specified temperature profiles and phase functions.

Nomenclature

A_ℓ = constants in phase function expansion
a_j = quadrature weight
B_j = defined by Eq. (37)
C_j, C_{m+1-j} = integration constants
$E(\mu)$ = defined by Eq. (30)
F = defined by Eq. (35)
I = monochromatic radiative intensity
I_b = Planck's blackbody intensity function
K_m, K_{m+1-j} = particular solution constants
m = order of Gaussian quadrature
N = number of terms in phase function expansion
n = refractive index
P_ℓ = Legrendre polynomial of degree ℓ
$p(\mu)$ = phase function
$Q(\mu)$ = defined by Eq. (29)
q_{0j} = defined in text
q_R = monochromatic radiative flux
$R_j(\mu)$ = defined by Eq. (28)
$S(\mu)$ = defined by Eq. (22)
v_j = unknown parameters, Eq. (7) and Eq. (15)
W = albedo parameter
x = matrix elements, Eqs. (7) and (8)
Y = defined in text
$Z(\mu)$ = defined by Eq. (27)
z_0 = defined by Eq. (40)
z_2 = defined by Eq. (44)
ξ = defined by Eq. (5)
λ_j = eigenvalues
μ = cosine of polar angle, Fig. 1
μ_i = quadrature points, $0 \leq \mu_i \leq 1$
τ = local monochromatic optical depth
τ_0 = monochromatic optical thickness

Subscripts
h = homogeneous solution
p = particular solution

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Introduction

THE integrodifferential equation describing one-dimensional, axisymmetric radiative transfer in an absorbing, emitting, and anisotropically scattering medium has the form[1]

$$\frac{\mathrm{d}I}{\mathrm{d}\tau}(\tau,\mu) = -\frac{I(\tau,\mu)}{\mu} + \frac{W}{2\mu}\int_{-1}^{1} I(\tau,\mu') \times \sum_{\ell=0}^{N} A_\ell P_\ell(\mu) P_\ell(\mu')\,\mathrm{d}\mu' + \frac{(1-W)n^2 I_b(\tau)}{\mu} \tag{1}$$

Here W is the scatter albedo, τ the optical depth, μ equals the cosine of the polar angle θ, n is the refractive index, $I_b(\tau)$ is Planck's blackbody intensity function, $P_\ell(\mu)$ is the Legendre polynomial of degree ℓ, of the first kind, and I is the radiative intensity (see Fig. 1). The constants A_ℓ are a result of expressing the phase function, $p(\varphi)$, as an N term series[2] of Legendre polynomials,

$$p(\varphi) = \sum_{\ell=0}^{N} A_\ell P_\ell(\cos\varphi)$$

with φ being the angle between the incident and scattered beams. One standard method of solution has been that of discrete ordinates whereby the integral term in Eq. (1) is approximated by a Gaussian quadrature.[2-4] The use of this approximation yields a system of differential equations expressed as

$$\frac{\mathrm{d}I}{\mathrm{d}\tau}(\tau,\mu_i) = -\frac{I(\tau,\mu_i)}{\mu_i} + \frac{W}{2\mu_i}\sum_{j=1}^{m} a_j I(\tau,\mu_j) \times \sum_{\ell=0}^{N} A_\ell P_\ell(\mu_j) P_\ell(\mu_i) + \frac{(1-W)n^2 I_b(\tau)}{\mu_i}, \quad i=1,\ldots,m \tag{2}$$

where μ_i are the quadrature points, a_j the quadrature weights, and m (an even integer) the quadrature order. The general solution to Eq. (2) is then given by

$$I(\tau,\mu_i) = I_h(\tau,\mu_i) + I_p(\tau,\mu_i), \quad i=1,\ldots,m \tag{3}$$

where $I_h(\tau,\mu_i)$ are the homogeneous solutions and $I_p(\tau,\mu_i)$ are the particular solutions. The homogeneous solutions of

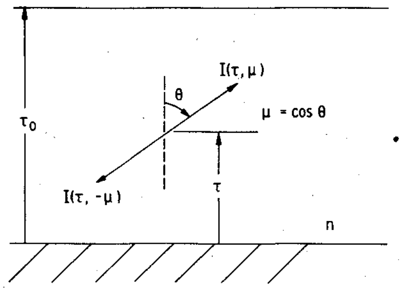

Fig. 1 Coordinate system and geometry.

Eq. (2) are already known.[2] In this paper, the objective is to determine the particular solution associated with Eq. (2) for $W \neq 1.0$ and also when an arbitrary function, $I_b(\tau)$, in Eq. (2) is specified. This is equivalent to having a specified but arbitrary temperature profile, $T(\tau)$, since I_b is a function of temperature, $I_b[T(\tau)]$. Knowing the homogeneous solutions of Eq. (2) yields the particular solutions by the method of variation of parameters.[5]

Analysis

In Ref. 2, the homogeneous solutions have been derived and may be expressed as

$$I_h(\tau,\mu_i) = \sum_{j=1}^{m/2} (1-\lambda_j\mu_j)\left[\frac{C_j e^{\lambda_j\tau}}{1+\mu_i\lambda_j} \times \sum_{\ell=0}^{N} A_\ell P_\ell(\mu_i)\xi_\ell(\lambda_j) + \frac{C_{m+1-j}e^{-\lambda_j\tau}}{1-\mu_i\lambda_j} \times \sum_{\ell=0}^{N} A_\ell P_\ell(\mu_i)\xi_\ell(-\lambda_j)\right] \tag{4}$$

where

$$\xi_0 = 1,\ \xi_1(\lambda_j) = (WA_0 - 1)/\lambda_j \tag{5a}$$

$$\xi_{\ell+1}(\lambda_j) = \xi_\ell(\lambda_j)\frac{WA_\ell - (2\ell+1)}{\lambda_j(\ell+1)} - \frac{\ell\xi_{\ell-1}(\lambda_j)}{(\ell+1)},\ \ell=1,\ldots,N-1 \tag{5b}$$

In Eq. (4), C_j and C_{m+1-j} are the m integration constants and λ_j are the $m/2$ positive eigenvalues which satisfy either of the relations

$$\frac{2}{W} = \sum_{k=1}^{m} \frac{a_k}{1+\mu_k\lambda_j} \sum_{l=0}^{N} A_\ell P_\ell(\mu_k)\xi_\ell(\lambda_j) \tag{6a}$$

$$\frac{2(1-W)}{W} = \sum_{k=1}^{m} \frac{a_k\mu_k\lambda_j}{1-\mu_k\lambda_j} \sum_{\ell=0}^{N} A_\ell P_\ell(\mu_k)\xi_\ell(-\lambda_j) \tag{6b}$$

By utilizing the variation of parameters method, the particular solutions $I_p(\tau,\mu_i)$ $[i=1,\ldots,m]$ are

$$\begin{bmatrix} I_p(\tau,\mu_1) \\ \vdots \\ I_p(\tau,-\mu_1) \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mm} \end{bmatrix} \begin{bmatrix} v_1(\tau) \\ \vdots \\ v_m(\tau) \end{bmatrix} \tag{7}$$

where the x elements are known via Eq. (4) and are given by

$$x_{i,j} = \frac{(1-\lambda_j\mu_j)C_j e^{\lambda_j\tau}}{1+\lambda_j\mu_i} \sum_{\ell=0}^{N} A_\ell P_\ell(\mu_i)\xi_\ell(\lambda_j) \tag{8a}$$

$$x_{i,m+1-j} = \frac{(1-\lambda_j\mu_j)C_{m+1-j}e^{-\lambda_j\tau}}{1-\lambda_j\mu_i} \times \sum_{\ell=0}^{N} A_\ell P_\ell(\mu_i)\xi_\ell(-\lambda_j)\quad i=1,\ldots,m;\ j=1,\ldots,m/2 \tag{8b}$$

and the parameters $v_j(\tau)$ $[j=1,\ldots,m]$ are to be determined from the system of differential equations

$$\begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mm} \end{bmatrix} \begin{bmatrix} \mathrm{d}v_1/\mathrm{d}\tau \\ \vdots \\ \mathrm{d}v_m/\mathrm{d}\tau \end{bmatrix} = (1-W)n^2 I_b(\tau) \begin{bmatrix} 1/\mu_1 \\ \vdots \\ 1/\mu_m \end{bmatrix} \tag{9}$$

where

$$\mu_k = -\mu_{m+1-k}\,[k=1,\ldots,m/2]$$

The solution of Eq. (9) for the elements

$$\frac{\mathrm{d}v_{m+1-k}}{\mathrm{d}\tau}\ [k=1,\ldots,m/2]$$

will have the form

$$\frac{\mathrm{d}v_{m+1-k}}{\mathrm{d}\tau} = \det\left[\frac{(1-\lambda_1\mu_1)C_1 e^{\lambda_1\tau}Y(\mu_1,\lambda_j)}{1+\lambda_1\mu_j},\ldots,\frac{1}{\mu_j},\ldots,\frac{(1-\lambda_1\mu_1)C_m e^{-\lambda_1\tau}Y(\mu_1,-\lambda_j)}{1-\lambda_1\mu_j}\right] \Big/ \det\left[\frac{(1-\lambda_1\mu_1)C_1 e^{\lambda_1\tau}Y(\mu_1,\lambda_j)}{1+\lambda_1\mu_j},\ldots,\frac{(1-\lambda_k\mu_k)C_{m+1-k}e^{-\lambda_k\tau}Y(\mu_k,-\lambda_j)}{1-\lambda_k\mu_j},\ldots,\frac{(1-\lambda_1\mu_1)C_m e^{-\lambda_1\tau}Y(\mu_1,-\lambda_j)}{1-\lambda_1\mu_j}\right] \times (1-W)n^2 I_b(\tau),\ k=1,\ldots,m/2 \tag{10}$$

where

$$Y(\mu_k,\lambda_j) = \sum_{\ell=0}^{N} A_\ell P_\ell(\mu_k)\xi_\ell(\lambda_j)$$

and det [...] indicates the determinant with the index j corresponding to the elements of each column. To obtain simplification, the following steps are carried out in both the numerator and denominator of Eq. (10):

1) Factor out $(1-\lambda_1\mu_1)$ from the first and last columns, $(1-\lambda_2\mu_2)$ from the second column and the next to last columns,..., and $(1-\lambda_{m/2}\mu_{m/2})$ from the $(m/2)$th and $(m/2+1)$th columns.

2) Factor out C_j from each column (i.e., factor C_1 from the first column, C_2 from the second column,...,C_m from the last column).

3) Factor out $e^{\lambda_j\tau}$ from the first $(m/2)$ columns and $e^{-\lambda_j\tau}$ from the last $(m/2)$ columns (i.e., factor $e^{\lambda_1\tau}$ from the first column,..., $e^{\lambda_{m/2}\tau}$ from the $(m/2)$th column, $e^{-\lambda_{m/2}\tau}$ from

the $(m/2+1)$th the column,..., and $e^{-\lambda_l\tau}$ from the last column). Execution of these steps yields

$$\frac{dv_{m+1-k}}{d\tau}=\frac{\prod_{i=1}^{m}C_i\prod_{i=1}^{m/2}(1-\lambda_i\mu_i)\prod_{i=1}^{m/2}e^{-\lambda_i\tau}\prod_{i=1}^{m/2}e^{\lambda_i\tau}(1-W)n^2I_b(\tau)K_{m+1-k}}{\prod_{i=1}^{m}C_i\prod_{i=1}^{m/2}(1-\lambda_i\mu_i)\prod_{i=1}^{m/2}e^{-\lambda_i\tau}\prod_{i=1}^{m/2}e^{\lambda_i\tau}C_{m+1-k}e^{-\lambda_k\tau}(1-\lambda_k\mu_k)},\ k=1,...,m/2 \tag{11}$$

where

$$K_{m+1-k}=\frac{\det\left[\frac{Y(\mu_l,\lambda_j)}{1+\lambda_l\mu_j},...,\frac{1}{\mu_j},...,\frac{Y(\mu_l,-\lambda_j)}{1-\lambda_l\mu_j}\right]}{\det\left[\frac{Y(\mu_l,\lambda_j)}{1+\lambda_l\mu_j},...,\frac{Y(\mu_k,-\lambda_j)}{1-\lambda_k\mu_j},...,\frac{Y(\mu_l,-\lambda_j)}{1-\lambda_l\mu_j}\right]} \tag{12}$$

Canceling like terms in both the numerator and denominator of Eq. (11) yields

$$\frac{dv_{m+1-k}}{d\tau}=\frac{e^{\lambda_k\tau}(1-W)n^2I_b(\tau)K_{m+1-k}}{C_{m+1-k}(1-\lambda_k\mu_k)},\ k=1,...,m/2 \tag{13}$$

Note that K_{m+1-k} depends only on the eigenvalues, quadrature points, and the constants $A_\ell(\ell=0,...,N)$. Proceeding using these steps, the expression for $dv_k/d\tau$ may be determined; therefore, in general form

$$\frac{dv_k}{d\tau}=\frac{K_k(1-W)n^2I_b(\tau)e^{-\lambda_k\tau}}{C_k(1-\lambda_k\mu_k)}\ k=1,...,m/2 \tag{14a}$$

$$\frac{dv_{m+1-k}}{d\tau}=\frac{K_{m+1-k}(1-W)n^2I_b(\tau)e^{\lambda_k\tau}}{C_{m+1-k}(1-\lambda_k\mu_k)}\quad k=1,...,m/2 \tag{14b}$$

Solving the differential equations in Eq. (14) yields

$$v_k=\frac{K_k(1-W)n^2}{C_k(1-\lambda_k\mu_k)}\int_{\tau_0}^{\tau}I_b(t)e^{-\lambda_k t}dt\quad k=1,...,m/2$$

$$v_{m+1-k}=\frac{K_{m+1-k}(1-W)n^2}{C_{m+1-k}(1-\lambda_k\mu_k)}\int_0^{\tau}I_b(t)e^{\lambda_k t}dt\quad k=1,...,m/2 \tag{15}$$

where τ_0 is the optical thickness.

Substituting the results from Eq. (15) back into Eq. (7) reveals the particular solutions as

$$I_p(\tau,\mu_i)=(1-W)n^2\left[\int_0^{\tau}I_b(t)\sum_{j=1}^{m/2}\{\frac{K_{m+1-j}e^{-\lambda_j(\tau-t)}}{1-\lambda_j\mu_i}\}dt\right.$$
$$\left.+\int_{\tau_0}^{\tau}I_b(t)\sum_{j=1}^{m/2}\{\frac{K_je^{-\lambda_j(t-\tau)}}{1+\lambda_j\mu_i}\}dt\right],i=1,...,m \tag{16}$$

If Eq. (16) is truly a solution of Eq. (2), then Eq. (16) must satisfy Eq. (2). For Eq. (16) to satisfy Eq. (2), K_j and K_{m+1-j} must satisfy the expression

$$\sum_{j=1}^{m/2}\{\frac{K_j}{1+\lambda_j\mu_i}\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(\lambda_j)+\frac{K_{m+1-j}}{1-\lambda_j\mu_i}$$
$$\times\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(-\lambda_j)\}=\frac{1}{\mu_i},i=1,...m \tag{17}$$

This is precisely the expression corresponding to Eq. (12), i.e., the K_j and K_{m+1-j} in Eq. (17) can readily be seen to satisfy an expression which has the form of Eq. (12). Thus it has been verified that Eq. (16) is a valid solution since K_j and K_{m+1-j} are required to satisfy the same expression both before and after substituting Eq. (16) into Eq. (2).

Determination of K_j and K_{m+1-j}

With the particular solutions given by Eq. (16), it would be especially convenient if the constants K_j and K_{m+1-j} $[j=1,...,m/2]$ could be determined. Similar to Ref. 6, it is possible to show that

$$K_j=-K_{m+1-j}\quad,\ j=1,...,m/2 \tag{18}$$

which means that Eq. (17) reduces to solving $m/2$ simultaneous linear nonhomogeneous algebraic equations instead of solving m equations. To show that Eq. (18) is true, it is necessary to use the following relationships

$$\sum_{\ell=0}^{N}A_\ell P_\ell(\mu)\xi_\ell(\lambda)=\sum_{\ell=0}^{N}A_\ell P_\ell(-\mu)\xi_\ell(-\lambda) \tag{19}$$

and

$$\sum_{\ell=0}^{N}A_\ell P_\ell(-\mu)\xi_\ell(\lambda)=\sum_{\ell=0}^{N}A_\ell P_\ell(\mu)\xi_\ell(-\lambda) \tag{20}$$

With the use of Eq. (5), the validity of Eqs. (19) and (20) can be easily be shown by separating the summation over N into even and odd terms.

To prove Eq. (18), is is necessary to first write the expression for K_j and K_{m+1-j} in the form of Eq. (12). Now if in the numerator of Eq. (12), the first and last rows are interchanged, the second and next to last rows are interchanged,..., and the $m/2$ th and $(m/2+1)$th rows are interchanged (along with appropriate sign changes), and then the first and last columns are interchanged, the second and next to last columns are interchanged,..., and the $m/2$th and $(m/2+1)$th columns are interchanged (along with appropriate sign changes), and Eq. (19) and (20) are applied, the result in Eq. (18) is readily obtained. Inserting Eq. (18) into Eq. (17) yields

$$\mu_i\sum_{j=1}^{m/2}K_j\left[\frac{1}{1+\lambda_j\mu_i}\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(\lambda_j)-\frac{1}{1-\lambda_j\mu_i}\right.$$
$$\left.\times\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(-\lambda_j)\right]=1,\ i=1,...,m/2 \tag{21}$$

where Eq. (21) supplies $m/2$ equations to be solved for the $m/2$ unknowns, K_j; the other $m/2$ unknowns are given by Eq. (18). At this point, the number of simultaneous algebraic equations to be solved has been reduced by a factor of 1/2. Further reduction of effort can be achieved if a modified version of the analysis in Ref. 2 entitled "Elimination of the Constants" is followed. This consists in writing Eq. (21) in the form

$$S(\mu)=\mu\sum_{j=1}^{m/2}K_j\Big[\frac{1}{1+\lambda_j\mu}\sum_{\ell=0}^{N}A_\ell P_\ell(\mu)\xi_\ell(\lambda_j)-\frac{1}{1-\lambda_j\mu}\sum_{\ell=0}^{N}(-1)^\ell A_\ell P_\ell(\mu)\xi_\ell(\lambda_j)\Big]^{-1} \tag{22}$$

which now replaces μ_i by μ thus allowing μ to be treated as a continuous variable instead of a fixed discrete value. Also $S(\mu_\alpha)=0$ $[\alpha=1,\ldots,m/2]$ in accordance with Eq. (21). From Eq. (5) it is noted that

$$\xi_\ell(-\lambda_j)=(-1)^\ell\xi_\ell(\lambda_j),\ell=0,1,\ldots,N \tag{23}$$

Finding a common denominator in Eq. (22) yields

$$S(\mu)=\mu^2\sum_{j=1}^{m/2}\sum_{\ell=0}^{N}\frac{K_jA_\ell P_\ell(\mu)\xi_\ell(\lambda_j)}{1-\lambda_j^2\mu^2}\times\Big[\frac{[1-(-1)^\ell]}{\mu}-\lambda_j[1+(-1)^\ell]\Big]^{-1} \tag{24}$$

which shows that $S(\mu)$ is an even function and may be expressed as $S(\mu)=S(\mu^2)$. If $S(\mu)$ is multiplied by

$$\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha{}^2\mu^2)$$

then

$$S(\mu)\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha{}^2\mu^2)$$

is a polynominal of degree at most $m+N'$ having roots μ_α $[\alpha=1,\ldots,m/2]$ where

$$N'=\begin{cases}N \text{ if } N \text{ even}\\ N-1 \text{ of } N \text{ odd}\end{cases} \tag{25}$$

Therefore it can be written that

$$S(\mu)\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha{}^2\mu^2)=Z(\mu)\prod_{\alpha=1}^{m/2}(\mu^2-\mu_\alpha{}^2) \tag{26}$$

where $Z(\mu)$ is a polynominal of order at most N'. Since the right-hand side of Eq. (26) must be an even function, then $Z(\mu)$ must hence be an even function of the form

$$Z(\mu)=z_0+z_2\mu^2+\ldots+z_{N'}\mu^{N'}=\sum_{\substack{k=0\\ \text{even}}}^{N'}z_k\mu^k \tag{27}$$

For simplification let $R_j(\mu)$, $Q(\mu)$, and $E(\mu)$ be defined as

$$R_j(\mu)=\prod_{\substack{\alpha=1\\ \alpha\neq j}}^{m/2}(1-\lambda_\alpha{}^2\mu^2) \tag{28}$$

$$Q(\mu)=\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha{}^2\mu^2) \tag{29}$$

and

$$E(\mu)=\sum_{\alpha=1}^{m/2}(\mu^2-\mu_\alpha{}^2) \tag{30}$$

Thus if $S(\mu)$ from Eq. (24) is substituted into Eq. (26), the result becomes

$$Z(\mu)E(\mu)=S(\mu)Q(\mu) \tag{31}$$

or

$$Z(\mu)E(\mu)=\mu^2\sum_{j=1}^{m/2}\sum_{\ell=0}^{N}R_j(\mu)K_jA_\ell P_\ell(\mu)\xi_\ell(\lambda_j)\Big[\frac{[1-(-1)^\ell]}{\mu}-\lambda_j[1+(-1)^\ell]\Big]^{-}Q(\mu) \tag{32}$$

Now in Eq. (32), let $\mu=1/\lambda_k[k=1,\ldots,m/2]$. Then Eq. (32) reduces to

$$Z(1/\lambda_k)E(1/\lambda_k)=\frac{1}{\lambda_k{}^2}\sum_{\ell=0}^{N}R_k(1/\lambda_k)K_kA_\ell P_\ell(1/\lambda_k)\xi_\ell(1/\lambda_k)\times\Big[\lambda_k[1-(-1)^\ell]-\lambda_k[1+(-1)^\ell]\Big]\quad k=1,\ldots,m/2 \tag{33}$$

Solving Eq. (33) for K_k yields

$$K_k=\frac{-\lambda_kE(1/\lambda_k)Z(1/\lambda_k)}{2R_k(1/\lambda_k)F(1/\lambda_k)}\qquad k=1,\ldots,m/2 \tag{34}$$

where

$$F(1/\lambda_k)=\sum_{\ell=0}^{N}(-1)^\ell A_\ell P_\ell(1/\lambda_k)\xi_\ell(1/\lambda_k) \tag{35}$$

From Eq. (34) it is seen that K_k can be computed if $Z(1/\lambda_k)$ can be computed, since the functions $R_k(1/\lambda_k)$, $E(1/\lambda_k)$, and $F(1/\lambda_k)$ are all known; namely, Eqs. (28, 30, and 35), respectively.

To determine the function $Z(1/\lambda_k)$ in Eq. (34), it is necessary to specify the coefficient $z_0, z_2,\ldots,z_{N'}$ of Eq. (27). Hence the object will now be to find these constants. Once these constants are known, Eq. (34) can be employed to determine the $m/2$ values of K_j. Then Eq. (18) can be used to find the remaining $m/2$ values of K_{m+1-j}. For convenience, let Eq. (34) be expressed as

$$K_j=B_jZ(1/\lambda_j),\quad j=1,\ldots,m/2 \tag{36}$$

where

$$B_j=\frac{-\lambda_jE(1/\lambda_j)}{2R_j(1/\lambda_j)F(1/\lambda_j)}\quad ,j=1,\ldots,m/2 \tag{37}$$

Substituting Eq. (36) for K_j into Eq. (32) yields

$$\mu^2\sum_{j=1}^{m/2}\sum_{\ell=0}^{N}R_j(\mu)B_jZ(1/\lambda_j)A_\ell P_\ell(\mu)\xi_\ell(1/\lambda_j)\times\Big[\frac{[1-(-1)^\ell]}{\mu}-[1+(-1)^\ell]\lambda_j\Big]-Q(\mu)=Z(\mu)E(\mu) \tag{38}$$

and then replacing $Z(\mu)$ by Eq. (27) gives the result

$$\mu^2\sum_{j=1}^{m/2}\sum_{\ell=0}^{N}\sum_{\substack{k=0\\ even}}^{N'}R_j(\mu)B_j\frac{z_k}{\lambda_j^k}A_\ell P_\ell(\mu)\xi_\ell(\lambda_j)\Big[\frac{[1-(-1)^\ell]}{\mu}-[1+(-1)^\ell]\lambda_j\Big]-Q(\mu)=E(\mu)\sum_{\substack{k=0\\ even}}^{N'}z_k\mu^k \tag{39}$$

The constant z_0 can now be determined by letting $\mu = 0$ in Eq. (39) with the result

$$z_o = -1/E(0) = (-1)^{(m/2)-1} \prod_{\alpha=1}^{m/2} \mu_\alpha^2 \tag{40}$$

Thus there are now $N'/2$ unknown z_k remaining; $N'/2$ equations can be obtained by taking $N'/2$ derivatives of Eq. (39) with respect to μ^2 and then evaluating the derivative at $\mu = 0$. Here it is seen that instead of requiring the solution of $m/2$ simultaneous equations from Eq. (21) for K_j that the problem has been reduced to solving $N'/2$ simultaneous nonhomogeneous algebraic equations for z_k (k = even) from which the values of K_j are then determined using Eq. (34).

As shown in Ref. 2, it is required that $2m > 2N+1$, thus necessarily $m > N$. In Ref. 1, sample problems for nonemitting media were presented for $N=3$ and values of m ranging from 4 to 24. To include emission in these problems, it would only be necessary to determine the $K_j [j=1,...,m/2]$. Using the preceding technique, this could be accomplished by simply solving one equation ($N'/2$) for one unknown z_2 rather than requiring the solution of $m/2$ or 12 simultaneous algebraic equations. (Note the one equation required for determining z_2 is obtained by taking the first derivative of Eq. (39) with respect to μ^2 and then evaluating it at $\mu = 0$.)

In summary the values of $K_j = -K_{m+1-j} [j=1,...,m/2]$ are given by Eq. (34) with $F(1/\lambda_k)$ defined by Eq. (35), $E(1/\lambda_k)$ by Eq. (30), $R_k(1/\lambda_k)$ by Eq. (28), and $Z(1/\lambda_k)$ by Eq. (27). The coefficients in Eq. (27) are obtained by the solution of $N'/2$ simultaneous algebraic equations generated by taking $N'/2$ derivatives of Eq. (39) with respect to μ^2 and then evaluating at $\mu = 0$. This technique can be used to obtain closed-form solutions for anisotropic scattering for values of $N \leq 5$; the value $N=5$ corresponds to $N' = 4$ so it is necessary to solve $N'/2$ or 2 algebraic equations for the two unknown z_2 and z_4. Two simultaneous equations can easily be solved thus allowing the solution for $K_j [j=1,...,m/2]$ to be determined in closed form for $N \leq 5$. This is of importance since many phase functions can be adequately described with $N \leq 5$. When $N > 5$ more than 2 algebraic equations must be solved and a library computer routine will be required.

Computations of K_j and K_{m+1-j}

To demonstrate the usefulness of the method for determining K_j and K_{m+1-j} $[j=1,...,m/2]$, some sample calculations were performed for $W=0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99, 0.999; also computation was done for various orders of Gaussian quadrature. Some of the results are shown in Tables 1-3 for the following phase functions[1,7]:

$$p(\mu) = 1 + [1.6 P_1(\mu) - P_2(\mu) + 0.4 P_3(\mu)]/3 \tag{41}$$

$$p(\mu) = 1 + [1.6 P_1(\mu) - 6P_2(\mu) + 0.4 P_3(\mu)]/9 \tag{42}$$

Table 1[a] Values of $K_{m+1-j} = -K_j$ for the phase function $p(\mu) = 1 + [1.6P_1(\mu) - P_2(\mu) + 0.4 P_3(\mu)]/3$

j	W = 0.10	W = 0.50	W = 0.90	W = 0.99	W = 0.999
1	0.0008386	0.2231012	2.0039997	7.6931816	24.7726886
2	0.0014918	0.0020570	0.0006621	-0.0001605	-0.0075628
3	0.0020513	0.0062654	0.0023060	-0.0004738	-0.0254078
4	0.0025842	0.0106385	0.0048141	-0.0007027	-0.0502599
5	0.0031221	0.0144880	0.0081732	-0.0005340	-0.0792158
6	0.0036896	0.0178780	0.0124236	0.0004282	-0.1091382
7	0.0043129	0.0210985	0.0176926	0.0026666	-0.1368623
8	0.0050272	0.0244976	0.0242412	0.0067819	-0.1592724
9	0.0058859	0.0284819	0.0325299	0.0135615	-0.1732229
10	0.0069736	0.0335840	0.0433324	0.0241076	-0.1753546
11	0.0084326	0.0406008	0.0579636	0.0400889	-0.1618406
12	0.0105228	0.0509016	0.0788133	0.0643130	-0.1279576
13	0.0137870	0.0672550	0.1108129	0.1023061	-0.0668527
14	0.0196164	0.0966512	0.1664977	0.1678301	0.0354749
15	0.0330904	0.1644848	0.2915247	0.3109025	0.2278817
16	0.0999167	0.4992410	0.8971071	0.9834561	0.9627966

[a]32-point single Gaussian quadrature [m = 32]

Table 2[a] Values of $K_{m+1-j} = -K_j$ for the phase function $p(\mu) = 1 + [1.6 P_1(\mu) - 6 P_2(\mu) + 0.4 P_3(\mu)]/9$

j	W = 0.10	W = 0.50	W = 0.90	W = 0.99	W = 0.999
1	0.0006301	0.2112277	2.1222237	8.1542495	26.2576076
2	0.0013212	0.0036629	0.0005684	0.0001738	-0.0025116
3	0.0019636	0.0098468	0.0020070	0.0006480	-0.0086465
4	0.0025822	0.0147838	0.0042477	0.0015003	-0.0175743
5	0.0031884	0.0182574	0.0073151	0.0028991	-0.0285061
6	0.0037974	0.0209188	0.0112849	0.0050961	-0.0403833
7	0.0044352	0.0233645	0.0163127	0.0084486	-0.0518441
8	0.0051433	0.0260750	0.0226789	0.0134587	-0.0611893
9	0.0059835	0.0295038	0.0308557	0.0208414	-0.0663229
10	0.0070484	0.0341880	0.0416232	0.0316465	-0.0646560
11	0.0084857	0.0409102	0.0563003	0.0475043	-0.0529352
12	0.0105583	0.0510191	0.0772797	0.0711967	-0.0268329
13	0.0138094	0.0672625	0.1094950	0.1082411	0.0203626
14	0.0196297	0.0966115	0.1654792	0.1724192	0.1029314
15	0.0330974	0.1644428	0.2908790	0.3138092	0.2706180
16	0.0999189	0.4992239	0.8968857	0.9844521	0.9774419

[a]32-point single Gaussian quadrature [m = 32]

Table 3[a] Values of $K_{m+1-j} = -K_j$ for the Rayleigh phase function $p(\mu) = 1.0 + [P_2(\mu)]/2$

j	W = 0.1	W = 0.5	W = 0.9	W = 0.99	W = 0.999
1	0.0008121	0.1835546	2.1485548	8.5641111	27.669458
2	0.0014135	0.0028601	0.0013127	0.0012393	0.0012341
3	0.0019302	0.0082949	0.0044959	0.0041837	0.0041583
4	0.0024347	0.0135607	0.0091835	0.0084877	0.0084264
5	0.0029659	0.0182001	0.0151498	0.0139981	0.0138892
6	0.0035544	0.0225045	0.0221902	0.0206216	0.0204629
7	0.0042293	0.0267897	0.0301731	0.0283599	0.0281625
8	0.0050233	0.0312995	0.0391087	0.0373583	0.0371483
9	0.0059805	0.0362742	0.0492290	0.0479728	0.0477914
10	0.0071705	0.0420676	0.0611067	0.0608893	0.0607931
11	0.0087166	0.0493206	0.0758845	0.0773770	0.0774384
12	0.0108604	0.0592914	0.0958138	0.0998880	0.1002029
13	0.0141307	0.0746926	0.1257415	0.1337051	0.1344144
14	0.0199131	0.1025215	0.1782096	0.1925616	0.1939255
15	0.0332915	0.1682544	0.2990186	0.3267689	0.3294972
16	0.0999878	0.5005415	0.8996884	0.9889284	0.9978361

[a]32-point single Gaussian quadrature [m = 32]

$$p(\mu) = 1 + [P_2(\mu)]/2 \tag{43}$$

Here Eq. (41) corresponds to primary forward and side scatter, Eq. (42) corresponds to primary side scatter, and Eq. (43) corresponds to Rayleight scattering. For isotropic scatter ($N = 0$ or $p(\mu) = 1.0$), the closed-form solution of Eq. (21) and tabulated results are contained in Ref. 6.

It should be noted that for Eqs. (41) and (42), $N=3$, and for Eq. (43), $N=2$, but for all three equations $N'/2=1$. Therefore with z_0 known from Eq. (40), z_2 need only be computed from the derivative of Eq. (39) with respect to μ^2. The expression for z_2 is

$$z_2 = \frac{-z_o \sum_{j=1}^{m/2} \left[\lambda_j - \frac{1}{\mu^2} + q_{0j} B_j Z_0 \right]}{1 + z_0 \sum_{j=1}^{m/2} \frac{q_{0j} B_j}{\lambda_j^2}} \tag{44}$$

where

$$q_{0j} = -2\lambda_j A_0 + 2A_1 \xi_1(\lambda_j) + \lambda_j A_2 \xi_2(\lambda_j) - 3A_3 \xi_3(\lambda_j)$$

The constants A_0, A_1, A_2, and A_3 are the coefficients of the phase functions in Eqs. (41-43); considering Eq. (41) as an example yields $A_0 = 1$, $A_1 = 1.6/3$, $A_2 = -1/3$, $A_3 = 0.4/3$. The results presented in Tables 1-3 were obtained by using Eq. (34) and also by solving Eq. (21) via a library routine for systems of nonhomogeneous linear algebraic equations (Cholesky method[8]). Results obtained by both methods were in excellent agreement.

Intensity, Flux, and Flux Divergence

The particular solutions to the radiative transport equation for one-dimensional axisymmetric transport with anisotropic scattering are given by Eq. (16) with the constants K_j and K_{m+1-j} $[j=1,...,m/2]$ obtained from Eq. (34) and making note of Eq. (18). The eigenvalues to be used in Eqs. (16) and (34) are easily determined from Eqs. (6a) or (6b). Combining the particular [Eq. (16)] and homogeneous [Eq. (4)] solutions, the general solution for the radiative intensity, Eq. (3), may be expressed as

$$I(\tau,\mu_i)=\sum_{j=1}^{m/2}(1-\lambda_j\mu_j)\Big[\frac{C_je^{\lambda_j\tau}}{1+\lambda_j\mu_i}\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(\lambda_j)+\frac{C_{m+1-j}e^{-\lambda_j\tau}}{1-\lambda_j\mu_i}\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(-\lambda_j)\Big]$$
$$+(1-W)n^2\Big[\int_0^\tau I_b(t)\sum_{j=1}^{m/2}\Big[\frac{K_{m+1-j}e^{-\lambda_j(\tau-t)}}{1-\lambda_j\mu_i}\Big]\mathrm{d}t+\int_{\tau_0}^{\tau}I_b(t)\sum_{j=1}^{m/2}\Big[\frac{K_je^{-\lambda_j(t-\tau)}}{1+\lambda_j\mu_i}\Big]\mathrm{d}t\Big],\ i=1,...,m \tag{45}$$

where C_j and C_{m+1-j} $[j=1,...,m]$ are the constants of integration to be determined through use of the specific boundary conditions applied to the given problem. The general solution, Eq. (45) is valid for a medium having one-dimensional, axisymmetric, radiative transport with absorption, emission, and anisotropic scattering. Also the temperature profile across the medium may be arbitrary, but specified.

Using Eq. (45), an expression for the radiative flux and flux divergence may be obtained since the radiative intensity has been assumed as a axisymmetric. The radiative flux at local optical depth τ may be defined as

$$q_R(\tau)=2\pi\int_{-1}^{1}I(\tau,\mu)\mu\mathrm{d}\mu \tag{46}$$

or in terms of the Gaussian quadrature

$$q_R(\tau)=2\pi\sum_{i=1}^{m}I(\tau,\mu_i)\mu_i a_i \tag{47}$$

Substituting Eq. (45) into Eq. (47) and rearranging yields

$$q_R(\tau)=2\pi\sum_{j=1}^{m/2}(1-\lambda_j\mu_j)\Big[\frac{C_j}{\lambda_j}e^{\lambda_j\tau}\times\sum_{i=1}^{m}\frac{a_i\mu_i\lambda_j}{1+\lambda_j\mu_i}\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(\lambda_j)+\frac{C_{m+1-j}}{\lambda_j}e^{-\lambda_j\tau}\sum_{i=1}^{m}\frac{a_i\mu_i\lambda_j}{1-\lambda_j\mu_i}\times\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(-\lambda_j)\Big]$$
$$+2\pi(1-W)n^2\Big[\int_0^\tau I_b(t)\sum_{j=1}^{m/2}K_{m+1-j}e^{-\lambda_j(\tau-t)}\times\sum_{i=1}^{m}\frac{a_i\mu_i}{1-\lambda_j\mu_i}\mathrm{d}t+\int_{\tau_0}^{\tau}I_b(t)\sum_{j=1}^{m/2}K_je^{-\lambda_j(t-\tau)}\sum_{i=1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i}\mathrm{d}t\Big] \tag{48}$$

The summations over index i in Eq. (48) can be simplified by use of Eq. (6b) for the sums which are not under the integrals and by employing the expression

$$\sum_{i=1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i}=-\sum_{i=1}^{m}\frac{a_i\mu_i}{1-\lambda_j\mu_i}=\frac{-2}{\lambda_j}\sum_{i=1}^{m/2}\frac{a_i\mu_i^2\lambda_j^2}{1-\lambda_j^2\mu_i^2} \tag{49}$$

from Ref. 6 for the sums which are under the integrals. The expression for $q_R(\tau)$ becomes

$$q_R(\tau)=\frac{-4\pi(1-W)}{W}\sum_{j=1}^{m/2}\frac{(1-\lambda_j\mu_j)}{\lambda_j}\times\Big[C_je^{\lambda_j\tau}-C_{m+1-j}e^{-\lambda_j\tau}\Big]$$
$$+4\pi(1-W)n^2\Big[\int_0^\tau I_b(t)\sum_{j=1}^{m/2}\frac{K_{m+1-j}}{\lambda_j}e^{-\lambda_j(\tau-t)}\times\sum_{i=1}^{m/2}\frac{a_i\mu_i^2\lambda_j^2}{1-\lambda_j^2\mu_i^2}\mathrm{d}t-\int_{\tau_0}^{\tau}I_b(t)\sum_{j=1}^{m/2}\frac{K_j}{\lambda_j}e^{-\lambda_j(t-\tau)}\times\sum_{i=1}^{m/2}\frac{a_i\mu_i^2\lambda_j^2}{1-\lambda_j^2\mu_i^2}\mathrm{d}t\Big] \tag{50}$$

For isotropic scattering, Eq. (50) reduces to the expression presented in Ref. 6. The expression for the flux divergence for anisotropic scattering is obtained from Eq. (50) by using Liebnitz rule and making note of Eq. (18); the result is

$$\frac{\mathrm{d}q_R(\tau)}{\mathrm{d}\tau}=\frac{-4\pi(1-W)}{W}\sum_{j=1}^{m/2}(1-\lambda_j\mu_j)\Big[C_je^{\lambda_j\tau}+C_{m+1-j}e^{-\lambda_j\tau}\Big]$$
$$-4\pi(1-W)n^2\int_0^{\tau_0}I_b(t)\sum_{j=1}^{m/2}K_{m+1-j}e^{-\lambda_j(|\tau-t|)}\times\sum_{i=1}^{m/2}\frac{a_i\mu_i^2\lambda_j^2}{1-\lambda_j^2\mu_i^2}\mathrm{d}t$$
$$+8\pi(1-W)n^2I_b(\tau)\sum_{j=1}^{m/2}\frac{K_{m+1-j}}{\lambda_j}\times\sum_{i=1}^{m/2}\frac{a_i\mu_i^2\lambda_j^2}{1-\lambda_j^2\mu_i^2} \tag{51}$$

The flux divergence for isotropic scattering may be obtained from Eq. (51) by using Eq. (6b) (for isotropic scattering) and by use of Eq. (A2) from Appendix A. Using these expressions to evaluate sums over i in Eq. (51) and to evaluate the sum over j in the last term of Eq. (51), the flux divergence for "isotropic" scattering becomes

$$\frac{\mathrm{d}q_R(\tau)}{\mathrm{d}\tau}=\frac{-4\pi(1-W)}{W}\sum_{j=1}^{m/2}(1-\lambda_j\mu_j)\Big[C_je^{\lambda_j\tau}+C_{m+1-j}e^{-\lambda_j\tau}\Big]$$
$$-\frac{4\pi(1-W)^2}{W}n^2\int_0^{\tau_0}I_b(t)\sum_{j=1}^{m/2}K_{m+1-j}e^{-\lambda_j(|\tau-t|)}\mathrm{d}t+4\pi(1-W)n^2I_b(\tau) \tag{52}$$

Expressions for the limiting cases of optically thin and optically thick may also be easily obtained by using the same approximations as employed in Ref. 9.

Sample Problem

To demonstrate an application of the method discussed in this paper, a sample problem will now be presented. For this problem the temperature profiles are chosen such that the temperature averaged across the medium's optical thickness is 1500 K. The three temperature profiles selected are parabolic, linear, and isothermal. Three phase functions are also considered: isotropic, and, from Ref. 1, Eqs. (41) and (42). The

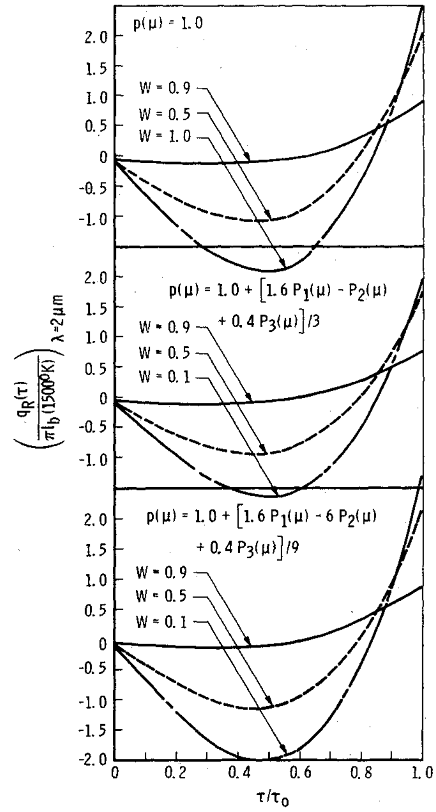

Fig. 2 Dimensionless monochromatic radiative flux for a coating ($n = 1.2$, $\tau_0 = 1.0$) on a gold substrate ($\bar{n} = 1.31 - i\ 10.70$) with temperature profile $T = [\{0.057 + (\tau/\tau_0)\}/\{2.2 \times 10^{-7}\}]^{1/2}$ K, $T_{average} = 1500$ K.

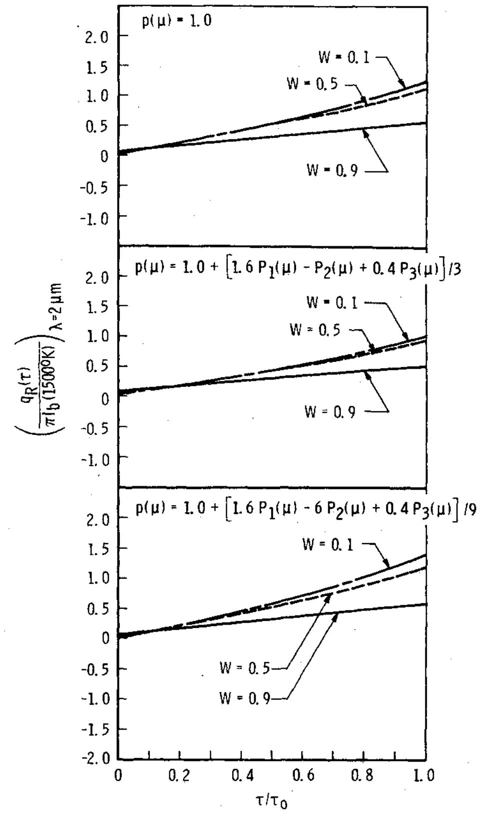

Fig. 4 Dimensionless monochromatic radiative flux for a coating ($n = 1.2$, $\tau_0 = 1.0$) on a gold substrate ($\bar{n} = 1.31 - i\ 10.70$) with temperature profile $T = 1500$ K.

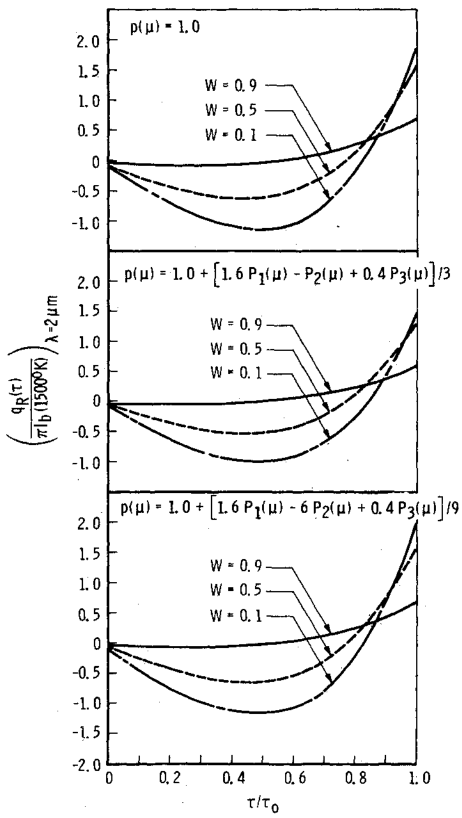

Fig. 3 Dimensionless monochromatic radiative flux for a coating ($n = 1.2$, $\tau_0 = 1.0$) on a gold substrate ($\bar{n} = 1.31 - i\ 10.70$) with temperature profile $T = [1000 + 1000\ (\tau/\tau_0)]$ K, $T_{average} = 1500$ K.

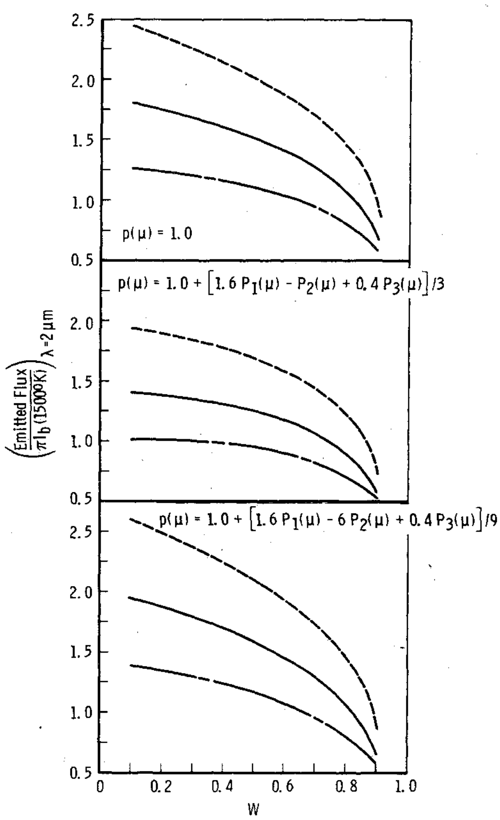

Fig. 5 Dimensionless monochromatic emitted flux leaving a coating ($n = 1.2$, $\tau_0 = 1.0$) on a gold substrate ($\bar{n} = 1.31 - i\ 10.70$) for various albedos and temperature profiles (------- $T = [\{0.057 + (\tau/\tau_0)\}/\{2.2 \times 10^{-7}\}]^{1/2}$ K, ——— $T = [1000 + 1000\ (\tau/\tau_0)]$ K, and — — — $T = 1500$ K; $T_{average} = 1500$ K).

participating medium consists of an absorbing, emitting, and scattering coating bounded by a gold substrate (refractive index[10] $\bar{n}=1.31-i\,10.70$) and at the top ($\tau=\tau_0$) by air; the optical thickness of the coating is taken to be unity ($\tau_0=1.0$). Both the substrate and top interface are assumed to be specular surfaces which reflect and transmit radiation in accordance with Fresnel's equations and Snell's law; the coating refractive index is 1.20.

Shown in Figs. 2-4 are the nondimensional radiative flux [Eq. (50)] profiles as a function of local optical thickness with albedo taken as a parameter for the three different temperature profiles. The radiative flux at any local optical thickness is nondimensionalized by the flux of a 1500 K blackbody in vacuum; all results are monochromatic at a wavelength[10] of 2μm. A 32-point single Gaussian quadrature was employed to obtain all of the results shown.

From Figs. 2 and 3 it is seen that near the center of the coating the net flux is lowest for the lowest albedo ($W=0.1$); Fig. 4 shows the isothermal case to have a monotonically increasing net flux. For all cases the flux at the top interface ($\tau=\tau_0$) is highest for $W=0.1$. This is as expected since the lowest albedo corresponds to greater emission. The side scattering phase function, Eq. (42), yields the highest net flux at $\tau=\tau_0$ for all temperature profiles; the forward and side scatter, Eq. (41), yields the lowest net flux at $\tau=\tau_0$. With respect to temperature profile, the one with the highest temperature near the top interface has the highest net flux at $\tau=\tau_0$. Here the parabolic profile has the highest temperature near the top interface and hence yields the highest net flux at $\tau=\tau_0$ followed by the linear and isothermal profiles.

Figure 5 shows the emitted flux leaving a coating as a function of albedo with phase function and temperature profile taken as parameters. This figure presents the really important heat transfer information. The parabolic profile produces the highest emitted flux leaving the coating with the side scattering phase function, Eq. (42), yielding the highest emission of the three phase functions. As expected the emitted flux decreases rapidly as the albedo increases.

Conclusion

Expressions for the radiative intensity [Eq. (45)], flux [Eq. (50)], and flux divergence [Eq. (51)], have been derived for the discrete ordinate form of the radiation transport equation. A medium having one-dimensional axisymmetric, radiative transfer with absorption, emission, and anisotropic scattering was assumed. Also the temperature profile across the medium could be arbitrary. Results were presented for a sample problem illustrating the influence of albedo, phase function, and temperature profile. The general solution presented is applicable to a wide range of problems involving radiative transfer in participating media. The method presented here is applicable to the same types of problems presented in Ref. 3. However the method presented in Ref. 3 required a large amount of numerical computation whereas the solution presented here allows the values $K_j=-K_{m+1-j}$ to be rapidly computed. The method presented here has the advantage of requiring at most $m/2$ values of $K_j[j=1,\ldots m/2]$ from Eq. (21). And, in fact, it has been shown that only $N'/2(N'<m)$ simultaneous equations are required to be solved. In addition to yielding an alternate approach to that of Ref. 3, the method presented here also permits obtaining simplified expressions for the radiative flux and flux divergence.

Appendix A: Derivation of an Identity

If Eq. (17) is multiplied by $a_i\mu_i$ and all m equations are added, the result is

$$\sum_{j=1}^{m/2}\left[\frac{K_j}{\lambda_j}\sum_{i=1}^{m}\frac{a_i\mu_i\lambda_j}{1+\lambda_j\mu_i}\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(1/\lambda_j)\right.$$

$$+\frac{K_{m+1-j}}{\lambda_j}\sum_{i=1}^{m}\frac{a_i\mu_i\lambda_j}{1-\lambda_j\mu_i}\times\left.\sum_{\ell=0}^{N}A_\ell P_\ell(\mu_i)\xi_\ell(-1/\lambda_j)\right]=2 \tag{A1}$$

Now using Eq. (6b) to evaluate the sums over index i, and Eq. (18) for simplification, the result becomes

$$-2\sum_{j=1}^{m/2}\frac{K_j}{\lambda_j}=\frac{W}{1-W} \tag{A2}$$

Appendix B: Optically Thin and Thick Approximations

Following the procedure outlined in Ref. 9, expressions for the limiting cases of optically thin and optically thick may be derived. To obtain an expression for the optically thin limit it is necessary to expand the exponential terms into a Taylor series expansion and retain only the first two terms. Then recalling that $\tau_0\ll 1.0(\tau_0\to 0)$, Eq. (50) becomes for the optically thin limit

$$q_R(\tau)=\frac{-4\pi(1-W)}{W}\left[\sum_{j=1}^{m/2}\frac{(1-\lambda_j\mu_j)}{\lambda_j}(C_j-C_{m+1-j})\right] \tag{B1}$$

and similarly the flux divergence from Eq. (51) becomes

$$\frac{dq_R(\tau)}{d\tau}=\frac{-4\pi(1-W)}{W}\sum_{j=1}^{m/2}(1-\lambda_j\mu_j)(C_j+C_{m+1-j})+8\pi(1-W)n^2I_b(\tau)\sum_{j=1}^{m/2}\frac{K_{m+1-j}}{\lambda_j}\times\sum_{i=1}^{m/2}\frac{a_i\mu_i^2\lambda_j^2}{1-\lambda_j^2\mu_i^2} \tag{B2}$$

while for isotropic scattering Eq. (52) becomes

$$\frac{dq_R(\tau)}{d\tau}=-4\pi(1-W)\left[\frac{1}{W}\sum_{j=1}^{m/2}(1-\lambda_j\mu_j)(C_j+C_{m+1-j})\times -n^2I_b(\tau)\right] \tag{B3}$$

The expression for the optically thick case is obtained for values of τ not near a boundary; it also becomes apparent that $C_j=0$ in Eq. (50). By expanding $I_b(t)$ in a Taylor series expansion and substituting in Eq. (50) and proceeding as in Ref. 9 the result becomes for the optically thick limit ($\tau_0\gg 1.0$ or $\tau_0\to\infty$)

$$q_R(\tau)=-8\pi\frac{(1-W)}{W}n^2\frac{dI_b(\tau)}{d\tau}\sum_{j=1}^{m/2}\frac{K_{m+1-j}}{\lambda_j^3}\times\sum_{i=1}^{m/2}\frac{a_i\mu_i^2\lambda_j^2}{1-\lambda_j^2\mu_i^2} \tag{B4}$$

For isotropic scattering, Eq. (B4) reduces to

$$q_R(\tau)=-8\pi\frac{(1-W)^2}{W}n^2\frac{dI_b(\tau)}{d\tau}\sum_{j=1}^{m/2}\frac{K_{m+1-j}}{\lambda_j^3} \tag{B5}$$

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